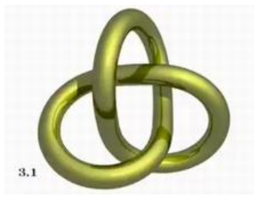
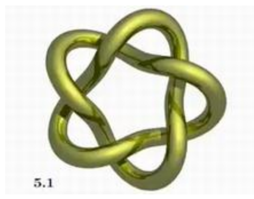
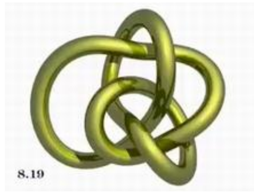
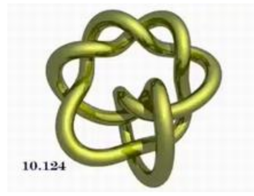
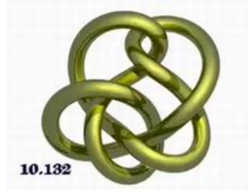
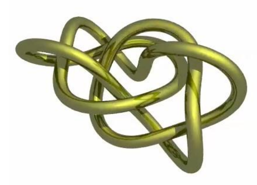
LORENZ & MODULAR FLOWS ARE KNOT SIMILAR

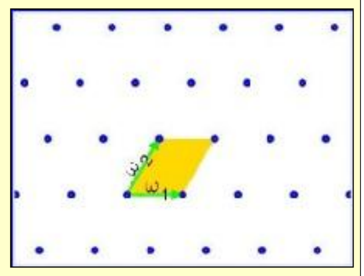
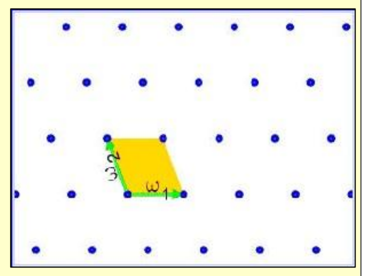
Overview

Knot theory (a branch of topology) can be used to show the connections between two very different fields of mathematics: (a) the chaotic dynamical systems found in weather systems whose solutions trace closed paths around strange attractors, and (b) modular flow dynamics on the space of 2-dimensional lattices.

The interplay between knot theory and dynamical systems helps uncover the universal laws governing the transition from regular to chaotic behavior.

Terminology

A knot, in the mathematical sense, is a closed, one-dimensional, non-self-intersecting curve that is embedded in three dimensions. Two mathematical knots are equivalent if one can be transformed into the other .

In number theory, a 2-D lattice is conceived as an infinitely large wire mesh which can be generated from a basis of linear combinations with integer coefficients. It consists of translations of a parallelogram. The lattice is considered normalized if the basic parallelogram has unit area.

Project Plan

The project will replicate the breakthrough work of Etienne Ghys (Knots and Dynamics, presented at International Congress of Mathematics, Madrid 2006) and Jos Leys (www.josleys.com) where they used graphical techniques to investigate knots that emerge in dynamical systems.

Hypothesis

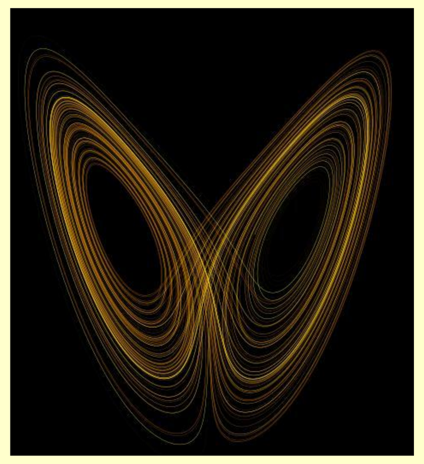
The hypothesis being tested is that a particular class of modular knots, pertaining to the theory of 2-dimensional lattices, is identical to the class of Lorenz knots pertaining to the theory of chaotic dynamical systems.

Background

Lorenz Knots:

In 1963, Edward Lorenz, mathematician and meteorologist at MIT, discovered chaos lurking in a simplified model for atmospheric convection. The model consisted of 3 non-linear ordinary differential equations:

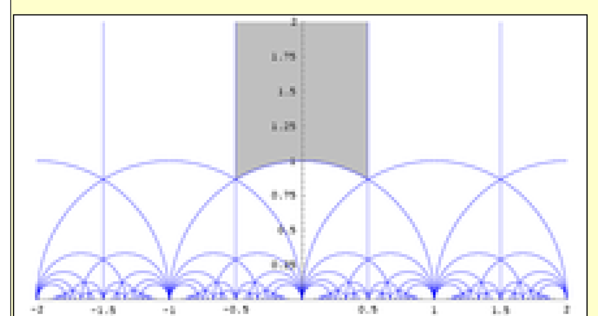
$\frac{dx}{dt}$=10(y-x) $\frac{dy}{dt}$=28x-y-xz $\frac{dz}{dt}$=xy - $\frac{8}{3}$z



The solution to these deterministic equations when plotted as trajectories in 3-dimensional space is concentrated into two broad approximately circular tracks that resemble butterfly wings. In 1982, Birman and Williams, showed that there existed infinite trajectories that were closed loops – these formed a variety of non trivial knots dubbed as Lorenz knots.

Modular Knots:

A normalized lattice can be represented by four real numbers – the coordinates of the two vertices. The two vertices can be treated as a 2x2 matrix with determinant 1. Multiplying the matrix with another matrix of determinant 1 gives us another normalized matrix. For a 2D lattice L lying on the real plane R^2 we define two complex numbers $g_2(L) = 60\sum_{\omega \varepsilon L\setminus\{0\}} \omega^{-4}$ and $g_3(L) = 140\sum_{\omega \varepsilon L\setminus\{0\}} \omega^{-6}$ where 60 and 140 are normalization coefficients. $g_2(L)$ and $g_3(L)$ are called Weierstrass functions which characterize the lattice if and only if the discriminant $g_2^3 - 27\, g_3^2$ is not zero. The lattices of area 1 are in the complement of the Trefoil Knot.



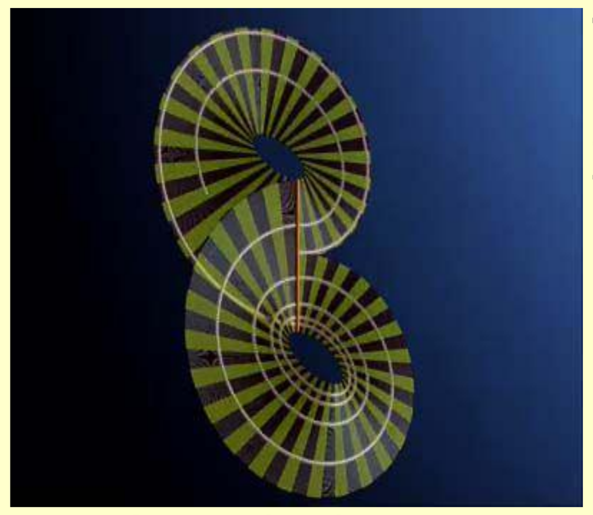
The adjacent image depicts a group of geodesics, which are curves whose tangent vectors remain parallel if they are transported along it. The shaded area is where all vectors with determinant one lie.

Methods

Programs were written in POVRay and Java to solve the non-linear ODE using a 4^{th}-order Runge Kutta and to generate the knot data. Some of the plots were made using UltraFractal.

Analysis

Lorenz knots can be recognized as prime and fibered. To make sense of the 3-dimensional Lorenz flow, we need to introduce the concept of the Lorenz template.



The Lorenz template is a 2D surface through which all Lorenz equation trajectories must travel. Birman and Williams showed that any Lorenz knot can be treated combinatorially by describing periodic orbits by an itinerary which consists of a finite sequence of symbols “L” and “R” which stand for travel in the Left or Right path respectively. From a knot-theoretic perspective, a closed path can be described by unique set of letters.

The group of all real matrices with determinant 1 is called as SL(2,R). When the components are all integers, then the corresponding matrices are called as SL(2,Z).

Returning to modular flows on 2D lattices, let us consider matrix M = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with integral coefficients and determinant 1. If M is hyperbolic (i.e. |a+d|>2), then there is a 2x2 matrix P such that $\Phi^t = P\begin{bmatrix} a & b \\ c & d \end{bmatrix}P^{-1} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ for some t.

Hence, for each integral matrix with determinant 1, there is for some t a fixed point, which is the periodic orbit of the modular flow. Each periodic orbit is a closed curve in the space of lattices of area 1, and defines a *modular knot* in the complement of the trefoil knot.

To show that modular knot is identical to a Lorenz knot, the following procedure can be used.

1. Find a template inside SL(2,R)/SL(2,Z) which looks like the Lorenz template.
2. Deform the lattices to make them approach the Lorenz template.

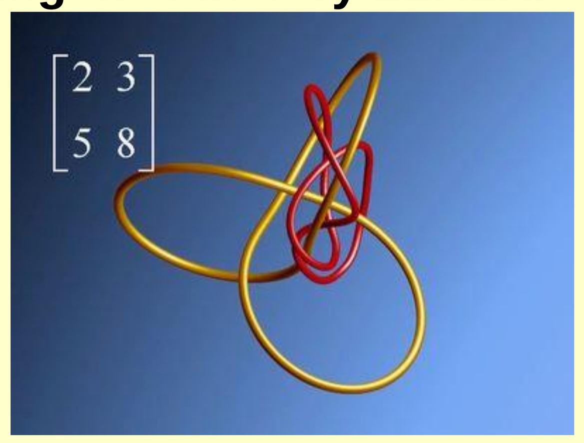
Using the proof by picture method, one can show that all modular knots are Lorenz knots. Proving the reverse proposition that every Lorenz knot is a modular knot is considerably more difficult.

Results

We will consider the case of the matrix $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ which is a normalized matrix with determinant 1. This matrix can be written as a product of matrices U= $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and V= $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

$\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$=UUVUV

We can plot the modular knot (in red) which will lie in the complement of the trefoil knot in yellow). For the particular case of $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$, the modular knot also happens to be a trefoil knot . Note that Birman and Williams also identified a trefoil knot as a Lorenz knot following the itinerary LLRLR.



Future Research

Since modular knots are much easier to generate than Lorenz knots, and yet preserve all the topological features of Lorenz knots, they could be used to better understand the strange attractor.

Conclusions & Discussion

Many theoretical and practical problems in mathematics, physics and biology are modeled by ordinary differential equations whose solutions are tied to the concept of flows. Understanding of flows requires study of instantaneous spatial properties as well as temporal behavior. These flows are solvable by applying both dynamical perspectives like chaos and entropy, as well as knot-theoretic tools.

Dynamics and knot theory are two exciting areas of mathematics that are tied together by the need to understand the topology of chaos. To quote Joan Birman, “the existence of small number of knots in a flow is like the onset of chaos”.