Modular Flows are in the complement of the Trefoil Knot

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Anita Rao

Texas Academy of Math & Science,

Denton, TX

Lattices and Matrices

A lattice L = $\left\{m \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + n \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} : m, n \text{ integers}\right\}$

consists of all integer linear combinations of two linearly independent vectors $\omega_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ & $\omega 2 = \begin{pmatrix} x_2 \\ y_1 \end{pmatrix}$

L can be seen as the image of matrix multiplication

$$\begin{pmatrix} m \\ n \end{pmatrix} \mapsto m \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + n \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

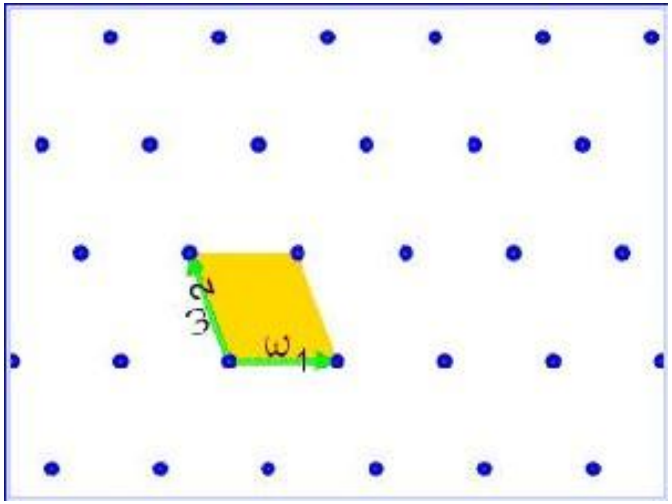
Lattices and Matrices

The same lattice L can be generated differently, e.g.

$$\begin{pmatrix} kx_1 + lx_2 & mx_1 + nx_2 \\ ky_1 + ly_2 & my_1 + ny_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} k & m \\ l & n \end{pmatrix}$$

when *k*, *l*, *m*, *n* are integers with $kn - lm = \pm 1$

The determinant of the matrix gives the area of the corresponding Parallelogram



Lattices and Matrices

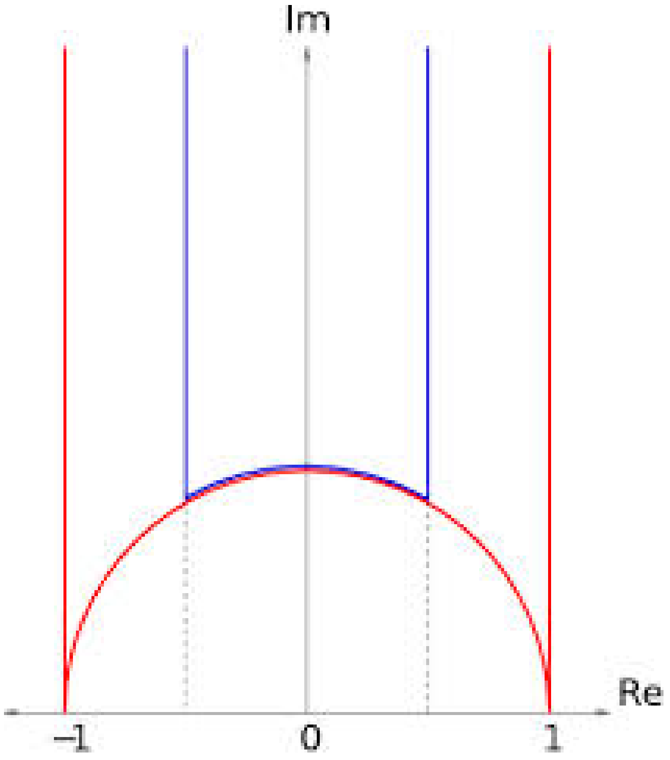
The group SL(2,R) consists of all real matrices of determinant 1, and its subset SL(2,Z) consists of all 2x2 integer matrices with determinant 1.

The quotient space SL(2,R)/SL(2,Z) consists of lattices whose fundamental parallelogram has area 1

If $\omega_i = x_i + iy_i$ *then* $Im(\omega_2/\omega_1) = \dfrac{x_1\, y_2 - x_2 y_1}{x_1^2 + y_1^2}$

Domain of SL(2,Z) & Modular Forms

Upper-half complex plane



Elliptic Functions

Eisenstein defined the series associated with a lattice L

$$E_k(\mathrm{L}) = \sum \frac{1}{\omega^k} \quad \text{for all } \omega \varepsilon L\backslash\{0\}$$

E_k(L) converges for all k ≥ 4 and
E_k(L) = 0 for all odd k.

Weierstrass showed that for every lattice, there exists a discriminant

$$\Delta(L) = 20\, E_4(\mathrm{L})^3 - 49\, E_6(\mathrm{L})^2$$

which is non-zero for every lattice

Space of All Lattices

Given any complex pair (a,b) so that

$\Delta(\mathrm{a},\mathrm{b}) = 20\ \mathrm{a}^3$ - 49 b^2 is non-zero, there is a unique lattice L such that (a,b) = (E_4(L), E_6(L))

The space of all lattices can be associated with the complex pairs (a,b) with $\Delta(\mathrm{a},\mathrm{b}) \neq 0$

$\mathrm{L} \mapsto (E_4(\mathrm{L}),\ E_6(\mathrm{L}))$

The space of lattices is identical to the quotient space GL(2,R)/GL(2,Z) which consists of:

$$(\boldsymbol{C}^2 \backslash \{\Delta = 0\}) = \{(a,b) : \Delta(a,b) \neq 0\}$$

Mapping from Lattice Space

Given a lattice L, there is a unique real number k so that (E_4(L), E_6(L)) lies on the "three-dimensional sphere"

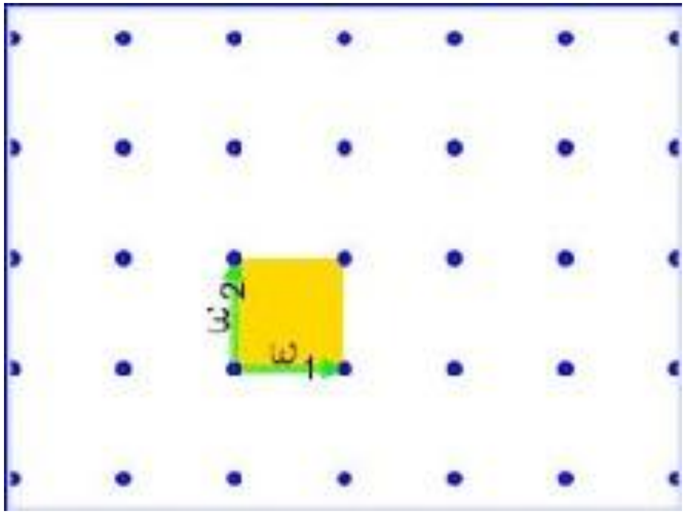
$$S^3 = \{(a, b) : |a|^2 + |b|^2 = 1\}$$

For a lattice of co-area 1, let us designate this number as k(L) with the mapping:

$$L \mapsto (E_4(\mathrm{k(L)L}), E_6(\mathrm{k(L)L}))$$

$$SL(2,R)/SL(2,Z) \cong (S^3 \backslash \{\Delta = 0\})$$
$$= \{(a,b) : |a|^2 + |b|^2 = 1 \; and \Delta(a,b) \neq 0\}$$

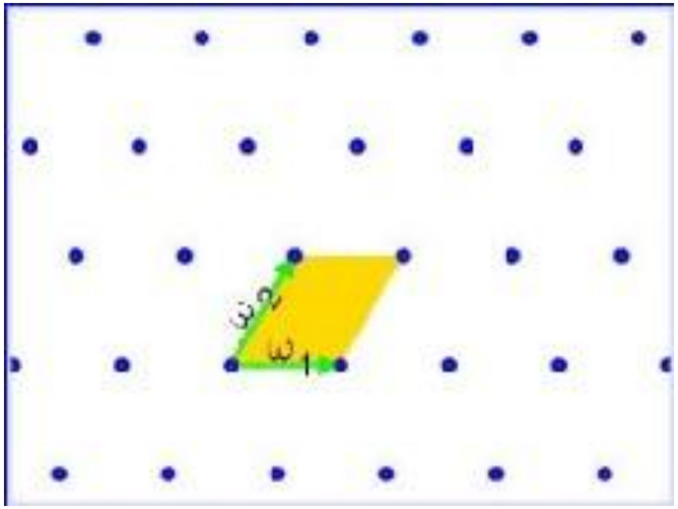
Special Cases of Lattices



L2 Square Lattice

E_6(L) = 0

Rotation by $\frac{\pi}{2}$ returns lattice to itself



L3 Hexagonal Lattice

E_4(L) =0

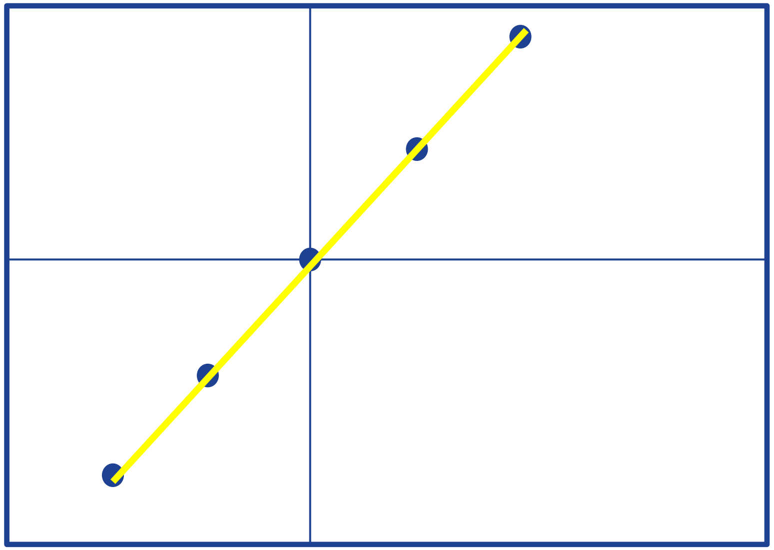
Rotation by $\frac{\pi}{3}$ returns lattice to itself

All other lattices need to be rotated by by π

Degenerate Lattice

$$E_4(\mathrm{L}) = \frac{\pi^4}{45}$$

$$E_6(\mathrm{L}) = \frac{2\pi^6}{945}$$



$$\Delta(L) = 20\, E_4(\mathrm{L})^3 - 49\, E_6(\mathrm{L})^2 = 0$$

A Dynamical System

Given a lattice L of co-area 1, there is natural way to shrink it in the y-direction and stretch it in the x-direction

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 & \alpha x_2 \\ \alpha^{-1} y_1 & \alpha^{-1} y_2 \end{pmatrix}$$

Since L = -L, attention can be restricted to positive α

By finding a real number t, so that $\alpha = e^t$ where t is a time parameter parametrizing the modular flow

L $\mapsto$ ϕ_t(L) in the space of lattices of co-area 1

This is a Dynamical System

Special Cases of Lattices

Given (a, b) in S^3, let L(a,b) denote the associated lattice of co-area 1 so that (a, b) = $(E_4$(k.L(a,b)L), E_6(k.L(a,b)L$)$ where k = k(L(a, b)).

At time t we send (a, b) to the image of L $\mapsto$ ϕ_t(L)

$$L \mapsto (E_4(\mathrm{k(L)L}), E_6(\mathrm{k(L)L}))$$

The resulting flow on S^3 is a called the modular dynamical system on S^3.